Statistical Shape Analysis on Elastic Surfaces using Numerical Inversion of SRNFs

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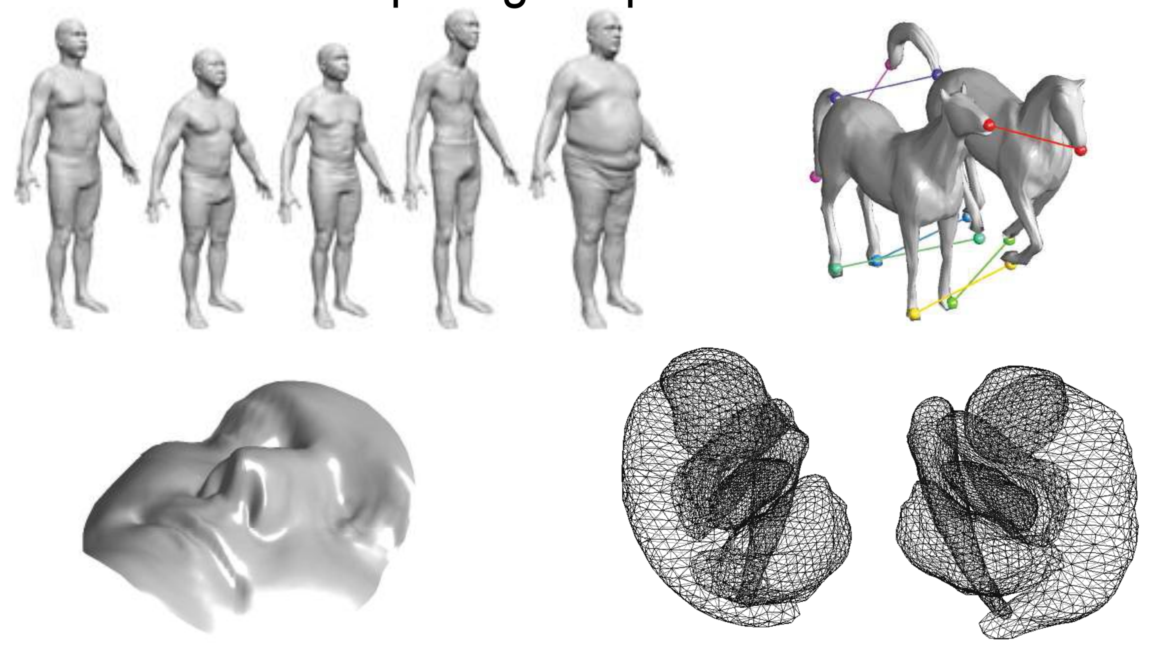
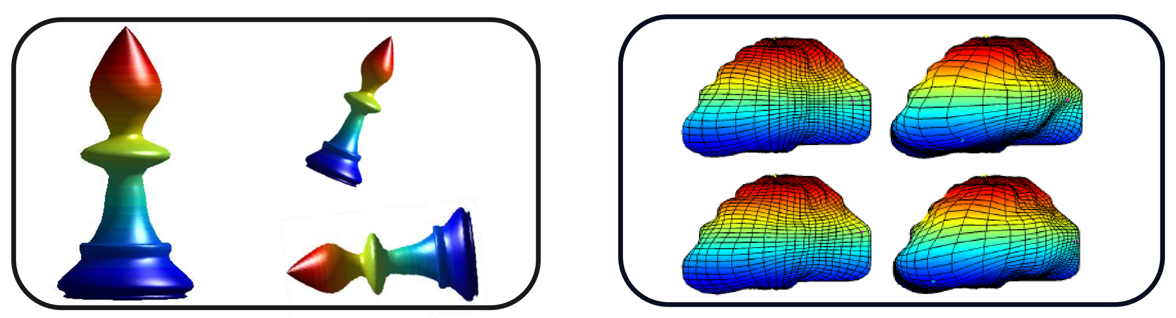
INTRODUCTION

- Goal: Many applications concern with capturing variability within and across shape classes
- Main focus is on statistical shape analysis in addition to comparing shapes



Shapes As Equivalence Classes

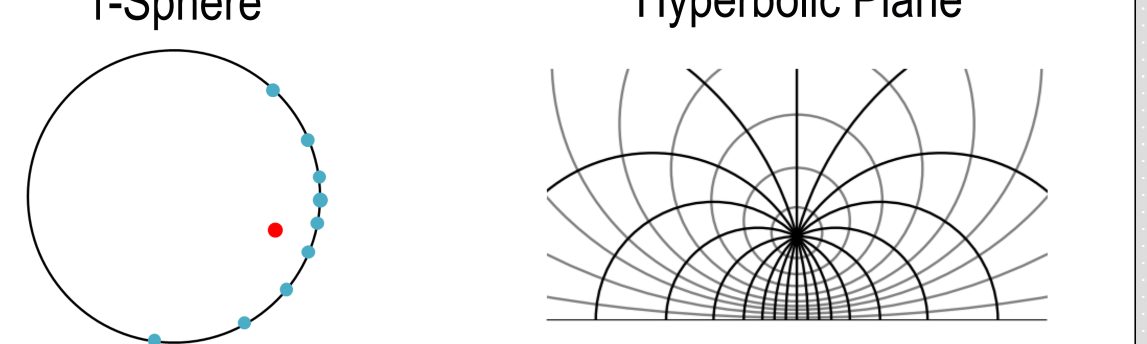
- Shape: geometrical information after modding out some desired invariances



Surfaces with the same shape: $f_1 \sim f_2$

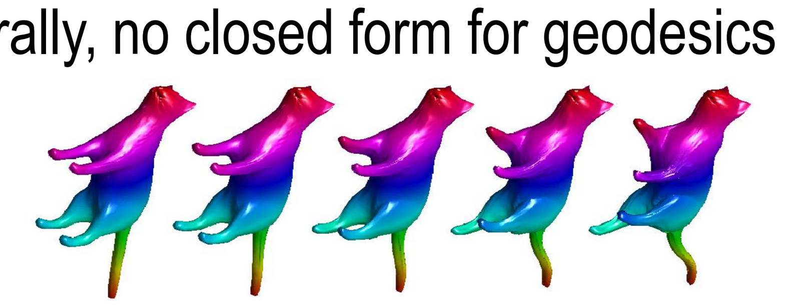
Shape Spaces Are Non-Euclidean

- Space of shapes, $S = \{[f] | f \in L^2\}$
- Shape spaces are non-Euclidean



Computing mean. 'Straight' lines and parallelism.

- Generally, no closed form for geodesics



A 'straight line' in the shape space.

SRNF of Surfaces

Definition. Given a spherical parameterized closed surfaces, $f: S^2 \to R^3$, its Square Root Normal Field (SRNF) is a mapping $f \mapsto Q(f)$, such that for all $s \in S^2$,

$$Q(f)(s) = \frac{n(s)}{|n(s)|^{1/2}}, \text{ where } n(s) = f_u(s) \times f_v(s).$$

Advantages of SRNF

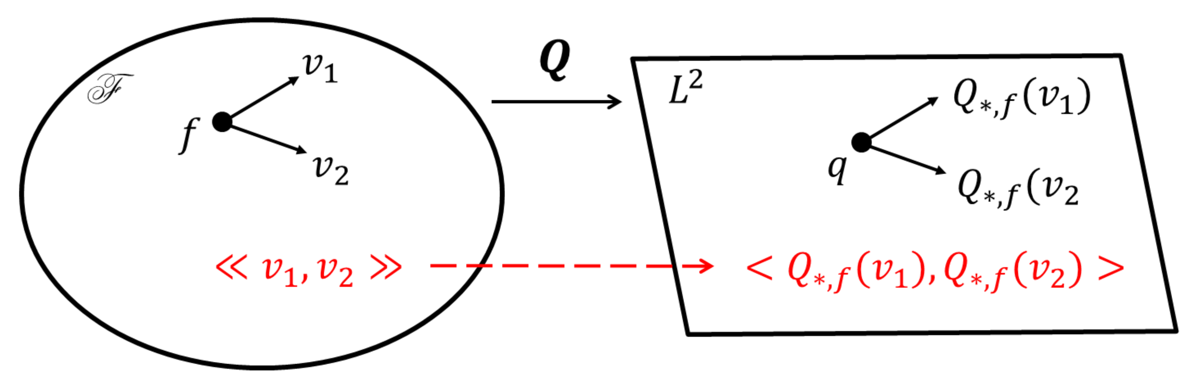
- The L^2 metric on F does not preserve distance,

$$\|f_1 - f_2\| \neq \|(f_1,\gamma) - (f_2,\gamma)\|$$

- The shape-invariant variables act on SRNFs by isometries,

$$\|q_1 - q_2\| = \|(q_1,\gamma) - (q_2,\gamma)\|$$

- The *elastic metric* becomes L^2 on SRNFs



- The *elastic metric* is physically interpretable

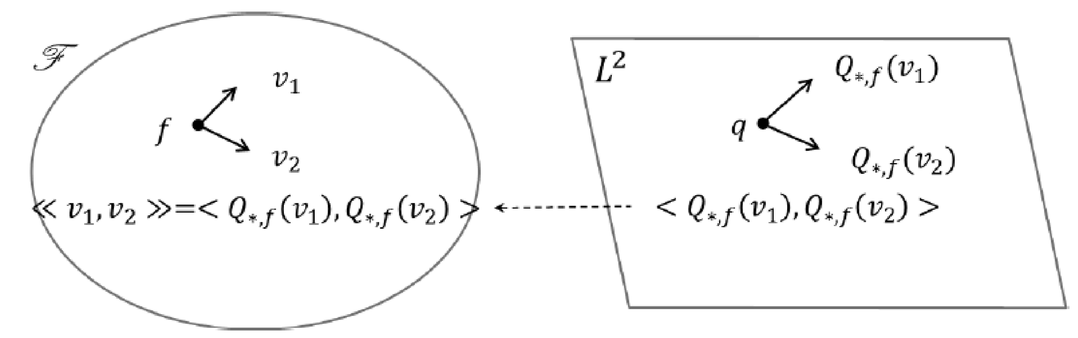
$$\frac{1}{4}\int_D \frac{\delta r_1(s)\delta r_2(s)}{r(s)}ds + \int_D \langle \delta\tilde{n}_1(s), \delta\tilde{n}_2(s)\rangle r(s)ds$$

Patch area — Normal direction

Proposed Framework

- In the absence of invertibility, some of the advantages of the SRNF are lost

Previous Calculation — **Proposed Calculation**



Numerically Complicated — Simplified Computation

Numerical Inversion by Optimization

- The energy function

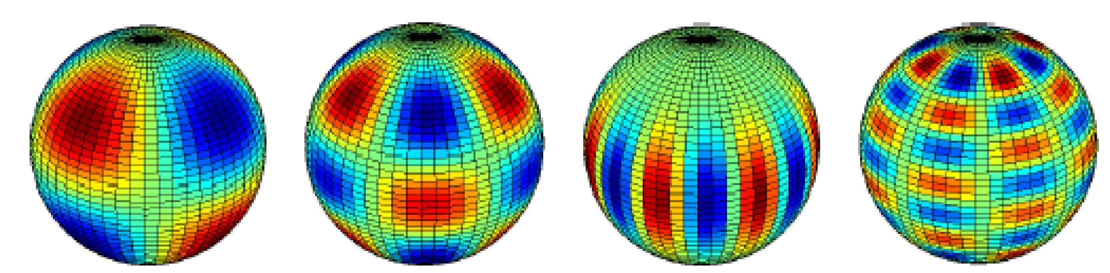
$$E(f;q) = \|Q(f) - q\|_2^2$$

- Given an SRNF, q_o, the original shape f_o s.t. $Q(f_o) = q_o$ is *estimated* as

$$f^* = argmin_{f \in F} E(f; q_o)$$

- Gradient descent

$$\nabla E(f;q) = \sum_{b \in B} \nabla_b E(f;q_o) \cdot b$$



Examples of Spherical Harmonic Basis

Simplified Calculation

- Simplified analysis
  - Geodesic computation (deformation)
  - Mean, PCA
  - Random sampling

Computing Mean

Previous

Algorithm 1 *Let μ_f^0 be an initial estimate. Set $j = 0$.*

1. *Register $f_1, \dots, f_n$ to μ_f^j.*
2. *For each $i = 1, \dots, n$, construct a geodesic to connect f_i to μ_f^j and evaluate $v_i = \exp^{-1}_{\mu_f^j}(q_i)$.*
3. *Compute the average direction $\bar{v} = \frac{1}{n}\sum_{i=1}^n v_i$.*
4. *If $\|\bar{v}\|$ is small, stop. Else, update $\mu_f^{j+1} = \exp_{\mu_f^j}(\epsilon\bar{v})$ by shooting a geodesic, ϵ¿0, small.*
5. *Set $j = j + 1$ and return to Step 1.*

n geodesics per iteration

Proposed

Algorithm 2 *Let $\bar{q} = Q(\mu_f^0)$ with μ_f^0 as an initial estimate. Set $j = 0$.*

1. *Register $Q(f_1), \dots, Q(f_n)$ to $\bar{q}$.*
2. *Update the average $\bar{q} = \frac{1}{n}\sum_{i=1}^n q_i$.*
3. *If change in $\|\bar{q}\|$ is small, stop. Else, set $j = j + 1$ and return to Step 1.*

Find μ_f by inversion s.t. $Q(\mu_f) = \bar{q}$.

1 inversion

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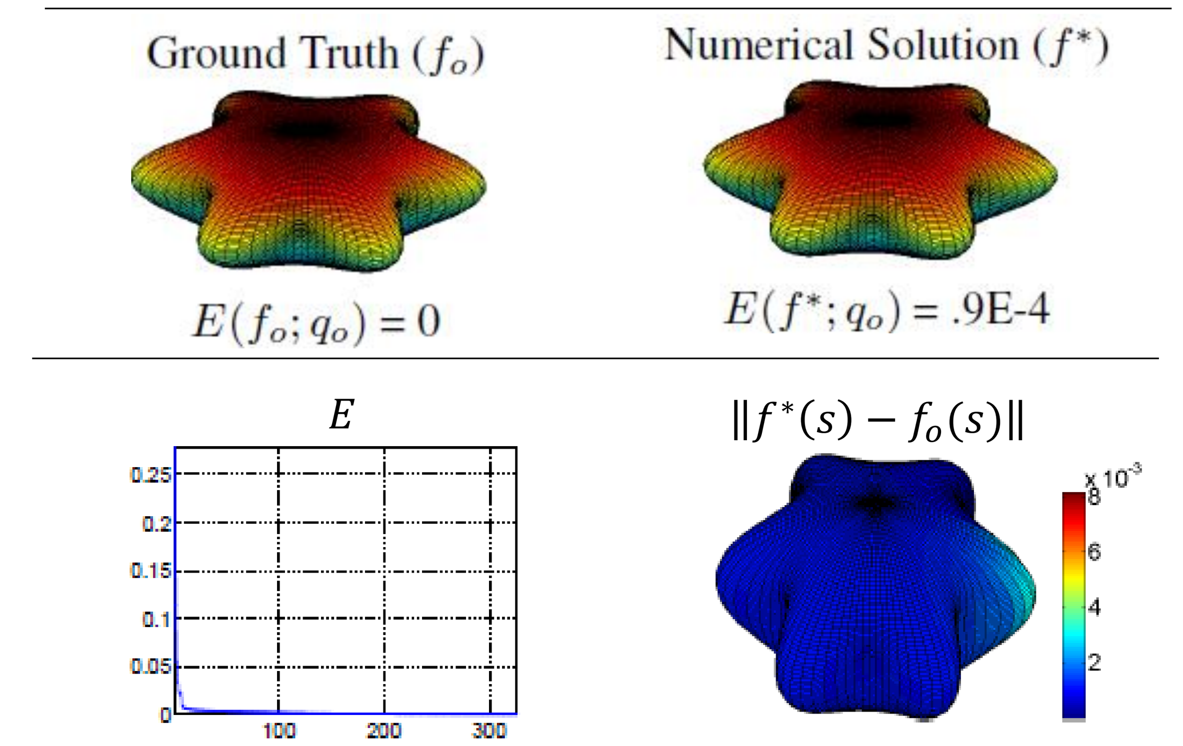
- S. Kurtek, E. Klassen, J.C. Gore, Z. Ding and A. Srivastava. Elastic Geodesic Paths in Shape Space of Parameterized Surfaces. PAMI, 34(9):1717-1730, 2012.

Approximated Inversion Map

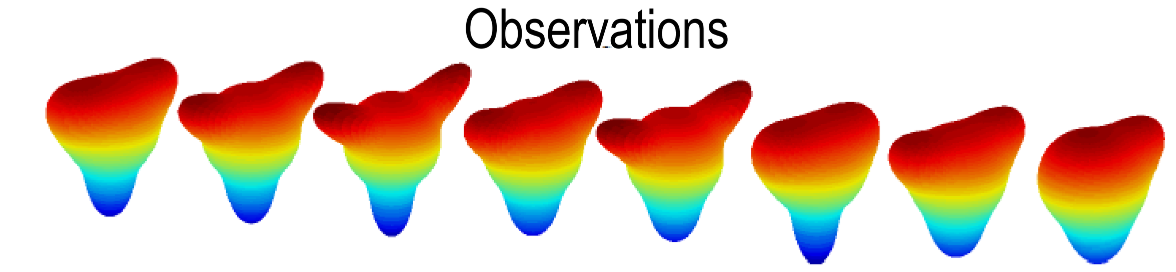
- Given q_o, find f^* to approximate the ground truth f_o, s.t. $Q(f_o) = q_o$.



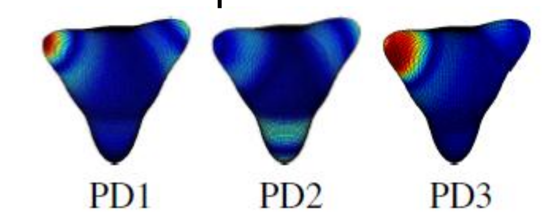
Statistical Analysis

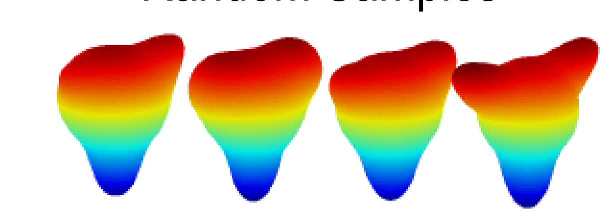
- Given a sample of observed surfaces $f_1, \dots, f_n$. Estimate the mean shape and principal directions of variation

Observations



Principal Directions — Random Samples





ADHD Classification

- Using shapes of subcortical structures in human brain to detect and quantify presence of the attention deficit hyperactivity disorder (ADHD)
- 19 with ADHD and 15 non-ADHD
- Gaussian classifier on principal components

Method Structure (%)	SRNF Gauss **Proposed**	SRM Gauss [29]	SRM NN [19]	Harmonic	ICP [5]	SPHARM PDM [13]
L. Caudate	67.7	-	41.2	64.7	32.4	61.8
L. Pallidus	85.3	88.2	76.5	79.4	67.7	44.1
L. Putamen	94.1	82.4	82.4	70.6	61.8	50.0
L. Thalamus	67.7	-	58.8	67.7	35.5	52.9
R. Caudate	55.9	-	50.0	44.1	50.0	70.6
R. Pallidus	76.5	67.6	61.8	67.7	55.9	52.9
R. Putamen	67.7	82.4	67.7	55.9	47.2	55.9
R. Thalamus	67.7	-	58.8	52.9	64.7	64.7